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Number of Printed Pages—8 **CS–504**

B. TECH

FIFTH SEMESTER EXAMINATION, 2002-2003

DISCRETE STRUCTURE

Time : Three Hours *Total Marks : 100*

Note :— Attempt **ALL** questions.

1. Attempt any FOUR of the following :— (5×4 =20)

(*a*) Let $A = R \times R$ (R is set of real numbers) and define the following relation on $A : (a, b)\, R\, (c, d)$ iff $a^2 + b^2 = c^2 + d^2$.

(*i*) Verify that (A, R) is an equivalence relation.

(*ii*) Describe geometrically what the equivalence classes are for this relation (Justify).

(*b*) For an integer $m \geq 3$, the graph T_m is defined as follows :—

The vertex set is the collection of all 2-subsets of an m-set (so T_m has $\binom{m}{2}$ vertices). And there is an edge between a pair of different vertices if and only if their intersection has exactly one element.

(*i*) Make a sketch of T_3, T_4 and T_5.

(*ii*) Which of T_3, T_4 and T_5 contains a Hamilton Cycle ?

(*iii*) For what $m \geq 3$ is T_m a bipartite graph ?

(iv) Formulate Euler's Theorem on the existence of an Euler tour in a graph. Show that for every $m \geq 3$, the graph T_m satisfies the conditions in Euler's Theorem.

(c) How many solutions are there for the equation

$$x_1 + x_2 + x_3 + x_4 = 20 \ ?$$

(i) If all x_i must be non-negative integers ;

(ii) If all x_i must be non-negative integers and x_4 is atmost 10.

(d) (i) Assume A, B and C are arbitrary sets. You don't have to prove anything or provide counter examples but only state if the following statements are True or False :—

(1) $\{ a, \varnothing \} \in \{a, \{a, \varnothing \}\}$.

(2) If $A \in B$ and $B \subseteq C$, then $A \in C$.

(3) If $A \in B$ and $B \subseteq C$, then $A \subseteq C$.

(4) If $A \subseteq B$ and $B \in C$, then $A \in C$.

(5) If $A \subseteq B$ and $B \in C$, then $A \subseteq C$.

(ii) Prove the following or provide a counter example :—

$$A \cup B \subseteq A \cap B \Rightarrow A = B$$

(e) (i) Let R be the binary relation defined as :

$$R = \{ < a, b > \in R^2 \mid a - b \leq 3 \}.$$

Determine whether R is reflexive, symmetric, anti-symmetric, and transitive.

(ii) How many distinct binary relations are there on the finite set A ?

(*f*) Let *f*, *g*, and $h : \Re \to \Re$: be defined by ($\Re$ is the set of real numbers) —

$$f(x) = x + 2, \quad g(x) = \frac{1}{x^2 + 1}, \quad h(x) = 3 \cdot$$

Compute :

(1) $f^{-1} \cdot g(x)$

(2) $h \cdot f \cdot \left(g \cdot f^{-1}\right) \cdot \left(h \cdot f(x)\right)$

2. Attempt any FOUR of the following :— (5×4 =20)

(*a*) Construct circuits for the following Boolean expressions :—

(*i*) $Q = A\overline{B} + \overline{A}B$

(*ii*) $Q = AA + AB + AC + BC$

(*b*) Fibonacci numbers F_n, are defined by —

$F_0 = 0,$

$F_1 = 1,$

$F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

Let the sequence of numbers G_n be defined by—

$G_1 = 1$

$G_2 = 3$

$G_n = G_{n-1} + G_{n-2} + 1$ for all $n \geq 3$.

Prove, using induction, that $G_n = 2\,F_n - 1$ for all $n \geq 1$.

(c) Prove in a Boolean algebra that if for some $X \in B$ we have $X = \bar{x}$, then this implies 0 =1.

(d) Let B be the set of all bit strings of a fixed length n. Let 0 be the string of n zeros and 1 be the string of n ones. Define + and . for arbitrary bit strings $a_1a_2 \cdots a_n$ and $b_1b_2 \cdots b_n$ as follows :—

$a_1a_2 \cdots a_n + b_1b_2 \cdots b_n = c_1c_2 \cdots c_n$, where each $c_i = \max(a_i, b_i)$

$a_1a_2 \cdots a_n \cdot b_1b_2 \cdots b_n = d_1d_2 \cdots d_n$, where each $d_i = \min(a_i, b_i)$

$\overline{a_1 a_2 \ldots\ldots a_n} = \bar{a}_1 \bar{a}_2 \ldots\ldots \bar{a}_n$, where $\bar{0}=1$ and $\bar{1}=0$.

Verify that $\langle B; 0, 1; +, ., ^{-} \rangle$ satisfies the following axioms of the definition of a Boolean Algebra :—

(i) $X + 0 = X$

(ii) $X\,\overline{X} = 0$

(e) For all integers, $n >= 2$, prove that —

$$\sum_{i=1}^{n-1} i\,(i+1) = (n\,(n-1)\,(n+1))/3$$

(f) Use induction to show that

$$2 + 4 + 6 + \ldots\ldots + 2n = n^2 + n$$

3. Attempt any FOUR of the following :— (5×4 =20)

(a) Consider the set $A = \{1, 2, 3, 4, 5\}$

Define the relation '<' on A such that $x < y$ if and only if (x mod 3) < (y mod 3).

(i) Prove that (A, < =) is a POSET.

(ii) Draw the Hasse diagram for (A, < =).

(iii) What are the maximal elements ?

(iv) What are the minimal elements ?

(*b*) Let $A \subseteq Z$ and $f : A \rightarrow N$ a one-to-one function. (Z is set of integers; *N* is set of Natural numbers.) We define a relation *R* on *A* as follows :—

for all $x ; y \in A$, $(x ; y) \in R$ if and only if $f(y) = k f(x)$ for some $k \in N$.

(*i*) Prove that *R* is a partial order relation on *A*.

(*ii*) let $A = \{ 4; 5; 6; 7; 8; 9 \}$ and suppose $f(x) = 10 - x$. Draw the Hasse diagram for the poset $(A; R)$.

(*c*) Suppose that the finite poset $(S, \leq)$ is a lattice. Prove that each finite lattice has a greatest element (i.e. there exists an element $\alpha \in S$ such that for all : $\beta \in S$, $\beta \leq \alpha$).

(*d*) Consider the partially ordered set $<\{ 2, 4, 6, 8 \}, | >$, where "$2|4$" means 2 divides 4.

Show with reason that whether the following statements are true or false :—

(*i*) Every pair of elements in the poset has a greatest lower bound.

(*ii*) Every pair of elements in the poset has a least upper bound.

(*iii*) This poset is a lattice.

(*e*) Given a Lattice $\langle L, \leq \rangle$, if the least upper bound of l_1 and l_2 is defined as $l_1 \vee l_2$ and greatest upper bound of l_1 and l_2 is defined by $l_1 \wedge l_2$.

Prove the following :—

$$(l_1 \vee l_2 = l_1) \Leftrightarrow (l_1 \wedge l_2 = l_2) \Leftrightarrow (l_2 \leq l_1).$$

(*f*) Prove that integers a and b can have at most one least common multiple.

4. Attempt any TWO of the following :— $(10 \times 2 = 20)$

(*a*) (*i*) Write the negation for the statement :

$$\forall x \in R,\ x > 3 \Rightarrow x^2 > 9$$

(*ii*) Write a negation for the following statement :—

$$\forall \text{ Sets } A, \text{ if } A \subseteq R \text{ then } A \subseteq Z$$

State which one is true – the statement or its negation. Explain your reasoning, where R is the set of real Numbers and Z is the set of integers. (5+5)

(*b*) Consider the following truth table :—

P	Q	R	S
1	1	1	0
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

(*i*) Construct a Boolean Expression having this table as its truth table. Simplify this expression. (4)

(*ii*) Construct a circuit having the given table as its input/output table. (6)

(c) The Peirce arrow $\downarrow$(NOR) is a logical binary operation which is defined as follows :—

$$p \downarrow q \equiv \sim (p \vee q)$$

(i) prove that $\sim p \equiv p \downarrow p$ (2)

(ii) prove that $p \wedge q \equiv (p \downarrow p) \downarrow (p \downarrow p)$ (3)

(iii) Write $p \rightarrow q$ using Peirce arrows only. (5)

5. Attempt any TWO of the following :— (10×2 =20)

(a) In a teaching room there are 20 seats, which are formed by 5 rows, each 4 tables wide. A certain class using that room has 9 students.

(i) In how many ways can these 9 students be seated in that teaching room ?

The teacher notes that in fact the students don't choose just any way to sit. There is a group of 5 students who always sit in the front two rows, such that 2 of them (the exact 2 can differ) sit in the front row, and the other 3 sit in the 2nd row. The other 4 students find seats in the final 3 rows.

(ii) In how many ways can the students be seated as described above, where the group of 5 students is predetermined ?

At another time a different class of 9 students is taught in this room. They always seat themselves so that there is atleast one student in each of the five rows.

(*iii*) Determine the number of ways that 9 students can be seated in the room so that there is at least one student in each of the five rows.

(*b*) Suppose the sequence (a_n), where $0 \leq n < \infty$, has generating function $f(x)$.

(*i*) What sequence is generated by the function $g(x) = (1+x)\, f(x)$?

(*ii*) And what sequence is generated by the function $h(x) = \frac{f(x)}{1+x}$?

(*c*) Prove using counting argument that —

(*i*) $\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+2}{k}$

(*ii*) $\binom{n}{m} = \sum_{k=0}^{r} \binom{r}{k}\binom{n-r}{m-k}$

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